Problem 2.25

For the configuration of Prob. 2.17, find the potential difference between a point on the axis and a point on the outer cylinder. Note that it is not necessary to commit yourself to a particular reference point, if you use Eq. 2.22.

Solution

An electrostatic field must satisfy $\nabla \times \mathbf{E} = \mathbf{0}$, which implies the existence of a potential function $-V$ that satisfies

$$\mathbf{E} = \nabla(-V) = -\nabla V.$$

The minus sign is arbitrary mathematically, but physically it indicates that a positive charge in an electric field moves from high-potential regions to low-potential regions (and vice-versa for a negative charge). To solve for V, integrate both sides along a path between two points in space with position vectors, $\mathbf{x}$ and $\mathbf{y}$, and use the fundamental theorem for gradients.

$$\begin{aligned}\int_{\mathbf{x}}^{\mathbf{y}} \mathbf{E} \cdot d\mathbf{l}_0 &= -\int_{\mathbf{x}}^{\mathbf{y}} \nabla V \cdot d\mathbf{l}_0 \\ &= -[V(\mathbf{y}) - V(\mathbf{x})] \\ &= V(\mathbf{x}) - V(\mathbf{y})\end{aligned}$$

According to Problem 2.17, the electric field around the coaxial cable is

$$\mathbf{E} = \begin{cases} \dfrac{\rho s}{2\epsilon_0}\hat{\mathbf{s}} & \text{if } s < a \\ \dfrac{\rho a^2}{2\epsilon_0 s}\hat{\mathbf{s}} & \text{if } a < s < b \\ \mathbf{0} & \text{if } s > b \end{cases}.$$

Since the electric field is radially symmetric, the path taken from $\mathbf{x}$ to $\mathbf{y}$ is a radial one and parameterized by s_0, where $x \le s_0 \le y$.

$$\int_x^y \mathbf{E}(s_0) \cdot d\mathbf{s}_0 = V(x) - V(y)$$

Since we want the potential difference between a point on the axis and a point on the outer cylinder, set $x = 0$ and $y = b$.

$$\begin{aligned}V(0) - V(b) &= \int_0^b \mathbf{E}(s_0) \cdot d\mathbf{s}_0 \\ &= \int_0^b [E(s_0)\hat{\mathbf{s}}_0] \cdot (\hat{\mathbf{s}}_0\, ds_0) \\ &= \int_0^b E(s_0)\, ds_0\end{aligned}$$

$V(0)$ is the electric potential on the axis, and $V(b)$ is the electric potential on the outer cylinder.

Plug in the electric field, evaluate the integrals, and simplify the result.

$$\begin{aligned}
V(0) - V(b) &= \int_0^a E(s_0)\, ds_0 + \int_a^b E(s_0)\, ds_0 \\
&= \int_0^a \frac{\rho s_0}{2\epsilon_0}\, ds_0 + \int_a^b \frac{\rho a^2}{2\epsilon_0 s_0}\, ds_0 \\
&= \frac{\rho}{2\epsilon_0}\left(\int_0^a s_0\, ds_0\right) + \frac{\rho a^2}{2\epsilon_0}\left(\int_a^b \frac{ds_0}{s_0}\right) \\
&= \frac{\rho}{2\epsilon_0}\left(\left.\frac{s_0^2}{2}\right|_0^a\right) + \frac{\rho a^2}{2\epsilon_0}\left(\left.\ln s_0\right|_a^b\right) \\
&= \frac{\rho}{2\epsilon_0}\left(\frac{a^2}{2}\right) + \frac{\rho a^2}{2\epsilon_0}(\ln b - \ln a) \\
&= \frac{\rho a^2}{2\epsilon_0}\left(\frac{1}{2} + \ln\frac{b}{a}\right)
\end{aligned}$$